

Solutions to Coursework #6, 2008

Geometry II

1 a) $\sigma_u \times \sigma_v = (-f_u, -f_v, 1)$

$$\|\sigma_u \times \sigma_v\|^2 = f_u^2 + f_v^2 + 1 > 0$$

so σ is regular

b) $N_\sigma = \frac{1}{\sqrt{f_u^2 + f_v^2 + 1}} (-f_u, -f_v, 1)$

$\frac{1}{\sqrt{f_u^2 + f_v^2 + 1}} > 0$ and so N_σ points upwards.

c) $N_\sigma = (0, 0, 1) \Leftrightarrow \frac{-f_u}{\sqrt{f_u^2 + f_v^2 + 1}} = 0$ and $\frac{-f_v}{\sqrt{f_u^2 + f_v^2 + 1}} = 0$ and $\frac{1}{\sqrt{f_u^2 + f_v^2 + 1}} = 1$

$$\Leftrightarrow f_u = 0 \text{ and } f_v = 0$$

2. a) No, it is missing the point $(0, 0, 0)$

b) If $f(u, v) = \sqrt{u^2 + v^2}$, then

$$f_u = \frac{u}{\sqrt{u^2 + v^2}}, \quad f_v = \frac{v}{\sqrt{u^2 + v^2}}$$

$$1 + f_u^2 + f_v^2 = 2$$

$$N_\sigma = \frac{1}{\sqrt{2}} \left(-\frac{u}{\sqrt{u^2 + v^2}}, -\frac{v}{\sqrt{u^2 + v^2}}, 1 \right)$$



c) We have $\sqrt{(r \cos \phi)^2 + (r \sin \phi)^2} = r$

$$\tilde{\sigma}_r = (\cos \phi, \sin \phi, 1)$$

$$\tilde{\sigma}_\phi = (-r \sin \phi, r \cos \phi, 0)$$

$$\tilde{\sigma}_r \times \tilde{\sigma}_\phi = (-r \cos \phi, -r \sin \phi, r)$$

$$\|\tilde{\sigma}_r \times \tilde{\sigma}_\phi\|^2 = r^2 \cos^2 \phi + r^2 \sin^2 \phi + r^2 = 2r^2 > 0$$

$\Rightarrow \tilde{\sigma}$ regular

$$N_{\tilde{\sigma}} = \frac{1}{\sqrt{2r^2}} (-r \cos \phi, -r \sin \phi, r)$$

$$= \frac{1}{\sqrt{2}} (-\cos \phi, -\sin \phi, 1)$$

$$= N_{\sigma} \text{ when } u = r \cos \phi, v = r \sin \phi$$

3d) $\phi(u, v) = (a + b \cos(u)) \cos(v), (a + b \cos(u)) \sin(v), b \sin(u)$

$$= (a \cos v, a \sin v, 0) + (b \cos(u) \cos v, b \cos(u) \sin v, b \sin(u))$$

(3)

~~$$(x, y) = (r, \phi) \quad \text{ie} \quad (u, v) = (r \cos \phi, r \sin \phi)$$~~

if the change of variables in Q1 (c).

~~$$\begin{aligned} \text{Then } \tilde{\sigma}_r \times \tilde{\sigma}_\phi &= (-r \cos \phi, -r \sin \phi, r) \\ &= \sqrt{u^2 + v^2} \left(\frac{-u}{\sqrt{u^2 + v^2}}, \frac{-v}{\sqrt{u^2 + v^2}}, 1 \right) \\ &= \sqrt{u^2 + v^2} \sigma_u \times \sigma_v \quad \text{using result of Q1} \end{aligned}$$~~

Meanwhile the theory predicts

~~$$\tilde{\sigma}_r \times \tilde{\sigma}_\phi = \det(J) \sigma_u \times \sigma_v$$~~

~~$$\det(J) = \frac{\partial u}{\partial r} \frac{\partial v}{\partial \phi} - \frac{\partial v}{\partial r} \frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial r} \frac{\partial v}{\partial \phi} - \frac{\partial v}{\partial r} \frac{\partial u}{\partial \phi}$$~~

~~$$= \cos \phi \cdot r \cos \phi - \sin \phi (-r \sin \phi) = r (\cos^2 \phi + \sin^2 \phi)$$~~

~~$$= r = \sqrt{u^2 + v^2}$$~~

as found.

Q3

(a)

$$\sigma_u = (-b \sin u \cos v, -b \sin u \sin v, b \cos u)$$

$$\sigma_v = ((a + b \cos u) (-\sin v), (a + b \cos u) \cos v, 0)$$

$$\sigma_u \times \sigma_v = b(a + b \cos u) \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin u \cos v & -\sin u \sin v & \cos u \\ -\sin v & \cos v & 0 \end{vmatrix}$$

$$= -b(a + b \cos u) (\cos u \cos v, \sin v \cos u, \sin u)$$

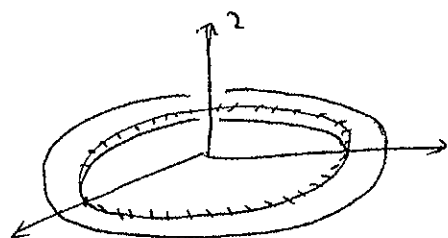
$$\text{So } N_\sigma = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = -(\cos u \cos v, \cos u \sin v, \sin u)$$

$$\text{as } \|\sigma_u \times \sigma_v\|^2 = b^2 (a + b \cos u)^2$$

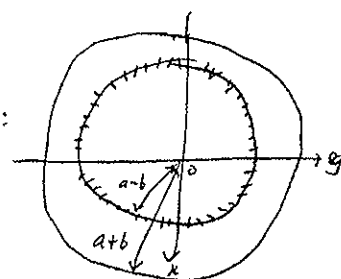
(4)

$$\begin{aligned}
 (b) \quad (i) \quad \gamma(t) &= \sigma(\overset{u=\pi}{\pi}, t) \\
 &= \left((a + \underbrace{b \cos \pi}_{-1}) \cos t, (a + \underbrace{b \cos \pi}_{-1}) \sin t, \cancel{b \sin \pi}_0 \right) \\
 &= (a-b) (\cos t, \sin t, 0)
 \end{aligned}$$

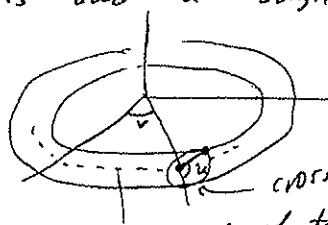
= circle in the $x-y$ plane radius $a-b$
 "inner rim"



or looking from on top:

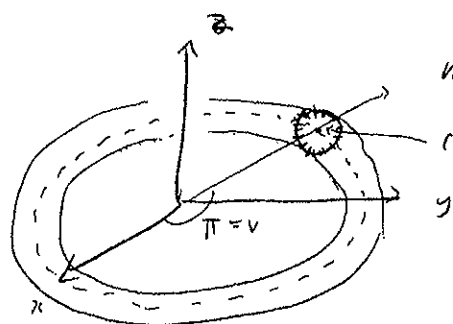


this fits the picture of $\sigma(u, v)$ where
 v = angle from the x -axis and u = angle lifted up off
 the $x-y$ plane:



cross-section = circle radius b
 midpoint of torus = circle radius a

$$\begin{aligned}
 (ii) \quad \gamma(t) &= \sigma(t, \pi) \\
 &= \left((a + \underbrace{b \cos \pi}_{-1}) \cos t, (a + \underbrace{b \cos \pi}_{-1}) \sin t, \cancel{b \sin t}_0 \right) \\
 &= -(a, 0, 0) + b (-\cos t, 0, \sin t)
 \end{aligned}$$



midpoint
 of curve
 centre is $(-a, 0, 0)$

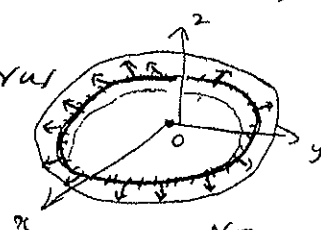
circle of radius b
 in the $x-z$ plane

$$N_{\sigma}(\pi, t) = -(\underbrace{\cos \pi}_{-1} \cos t, \underbrace{\cos \pi}_{-1} \sin t, \cancel{\sin \pi}_0) = (\cos t, \sin t, 0)$$

points radially outward into body of torus

in the $x-y$ plane. in

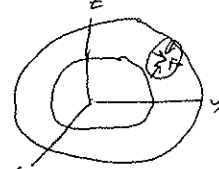
(case (i)).



In case (ii):

$$N_{\sigma}(t, \pi) = -(\cos t \cos \pi, \cos t \sin \pi, \sin t) = (\cos t, 0, -\sin t)$$

points radially inwards into small circle in $x-y$ plane



Q4 (a) $\tilde{\sigma}_{\tilde{u}} = (\sin \tilde{u} \cos \tilde{v}, -\cos \tilde{u}, \sin \tilde{u} \sin \tilde{v})$
 $\tilde{\sigma}_{\tilde{v}} = (\cos \tilde{u} \sin \tilde{v}, 0, -\cos \tilde{u} \cos \tilde{v})$

$$\begin{aligned} \tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}} &= \begin{pmatrix} \cos^2 \tilde{u} \cos \tilde{v} & \cos \tilde{u} \sin \tilde{u} (\cos^2 \tilde{v} + \sin^2 \tilde{v}) \\ \cos \tilde{u} \cos \tilde{v} & \sin \tilde{u} & \cos^2 \tilde{u} \sin \tilde{v} \\ \cos \tilde{u} \sin \tilde{v} & 0 & -\cos \tilde{u} \cos \tilde{v} \end{pmatrix} \\ &= \cos \tilde{u} \begin{pmatrix} \cos \tilde{u} \cos \tilde{v} & \sin \tilde{u} & \cos^2 \tilde{u} \sin \tilde{v} \\ \cos \tilde{u} \sin \tilde{v} & 0 & -\cos \tilde{u} \cos \tilde{v} \end{pmatrix} \\ &= -\cos \tilde{u} \tilde{\sigma}(\tilde{u}, \tilde{v}) \end{aligned}$$

$\therefore \|\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}\| = |\cos \tilde{u}|$ as $\tilde{\sigma}$ has unit length
 as $\tilde{u} \in (-\frac{\pi}{2}, \frac{\pi}{2})$
 $\therefore \frac{\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}}{\|\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}\|} = -\tilde{\sigma}(\tilde{u}, \tilde{v})$

(b) The sphere has an atlas $(\sigma, U), (\tilde{\sigma}, \tilde{U})$ with $N_{\sigma} = N_{\tilde{\sigma}}$ in the region of overlap (since in both cases point into centre of sphere) $\therefore$ orientable.

(c) My way of explaining was in lecture. (will accept anything that shows you understood).